Department of Geography and Earth Sciences

WRM 625

Hydrological Modeling

Lecture 6

Convex Routing Method

- Simplified procedure for routing hydrographs through stream reaches based on:

$$I - O = \frac{dS}{dt} \approx \frac{\Delta S}{\Delta t}$$

$$S_0 = b\left[\left(\frac{O}{a}\right)^{I/d}\right]^m = b\left(\frac{O}{a}\right)^{\frac{m}{d}} = b\left[\left(\frac{1}{a}\right)^{\frac{m}{d}}\right]O^{m/d} = KO^{m/d}$$

$$S = xS_1 + (1 - x)S_0$$

- Assumes m/d = 1 and x=0

Convex Routing

- Since m/d=1 and x=0, then $S = xKI^{m/d} + (1-x)KO^{m/d}$ reduces to:

$$S = KO$$

- If $I = I_1$ and $O = O_1$ in $I - O = \frac{dS}{dt} \approx \frac{\Delta S}{\Delta t}$ and substituting for S gives

$$I_1 - O_1 = \frac{K(O_2 - O_1)}{\Delta t}$$

- Rearranging and solving for the unknown O_2 gives $O_2 = I_1 \frac{\Delta t}{K} - O_1 \frac{\Delta t}{K} + O_1$ or $O_2 = I_1 \frac{\Delta t}{K} + O_1(1 - \frac{\Delta t}{K})$

Convex Routing

- Suppose $C = \frac{\Delta t}{K}$ in $O_2 = I_1 \frac{\Delta t}{K} + O_1 (1 - \frac{\Delta t}{K})$, then

$$O_2 = CI_1 + O_1(1 - C)$$

- For analysis and synthesis, the upstream hydrograph is known in both cases.
- For analysis, the downstream hydrograph is also known, but the routing coefficient C is not known.

Example

- Suppose the routing coefficient of a stream reach is 0.3 and I_t = 25 and O_t = 13 for t=0 and the next I_t values are 28,33, and 41, compute the next O_t value .
- Solution: The routing equation is

$$O_{t+\Delta t} = 0.3 I_t + 0.7 O_t$$

I	O
25	13
28	17=0.3(25)+0.7(13)
33	20=0.3(28)+0.7(17)
41	24=0.3(33)+0.7(20)

- Assigned task: 4 methods for estimating C (Mc Cuen)

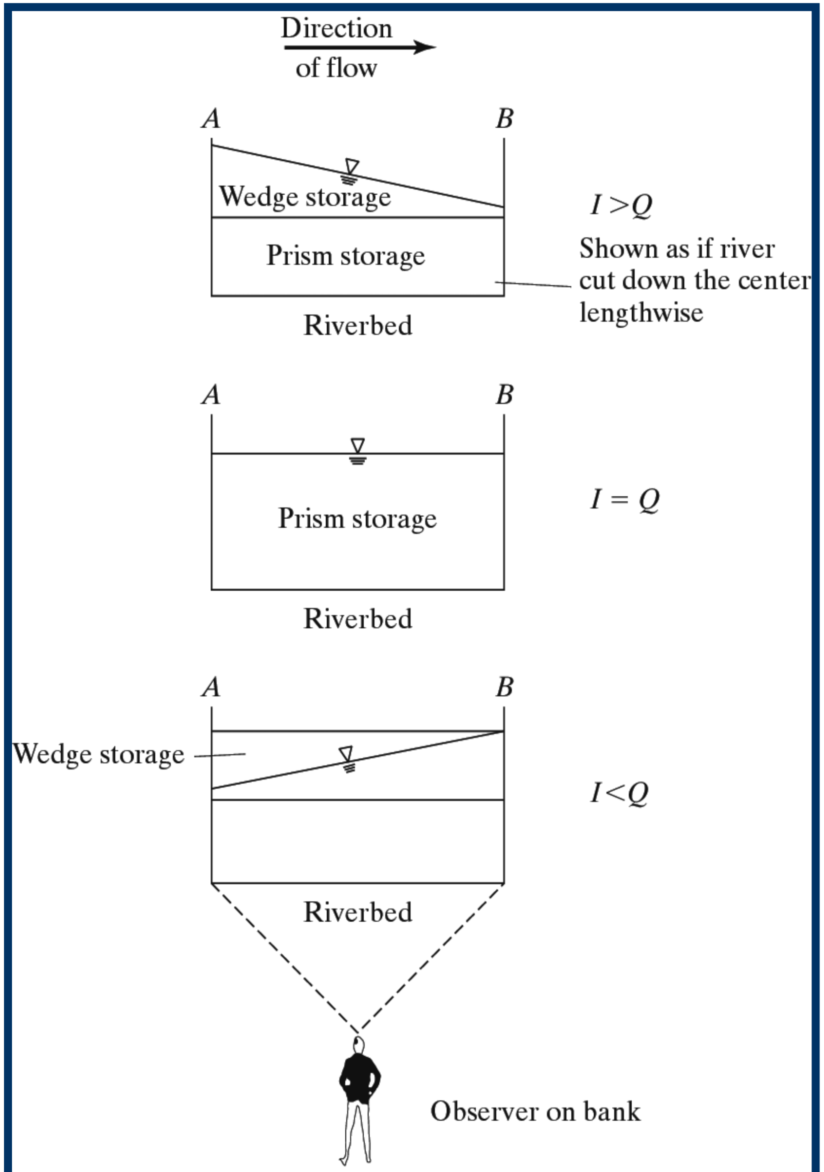


Figure 4.3

Prism and wedge storage concepts.

Wedge and Prism Storage

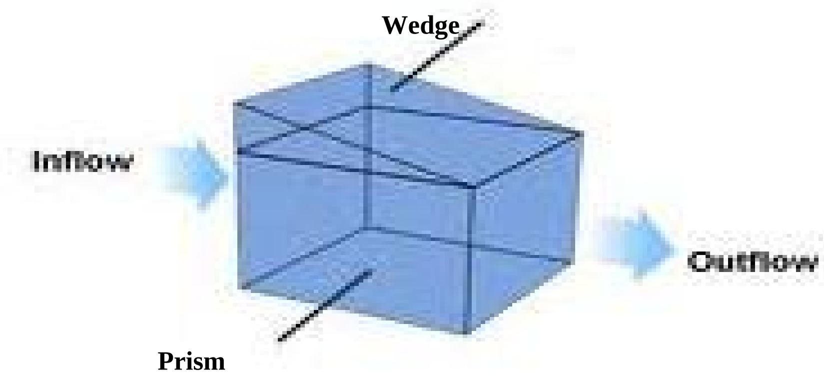
- **Positive wedge** $I > Q$
- **Maximum S when** $I = Q$
- **Negative wedge** $I < Q$

Muskingum Method

$S_p = K\,O$ Prism Storage

$S_w = K(I - O)X$ Wedge Storage

$S = K[XI + (1-X)O]$ Combined



Hydrologic river routing (Muskingum Method)

Wedge storage in reach

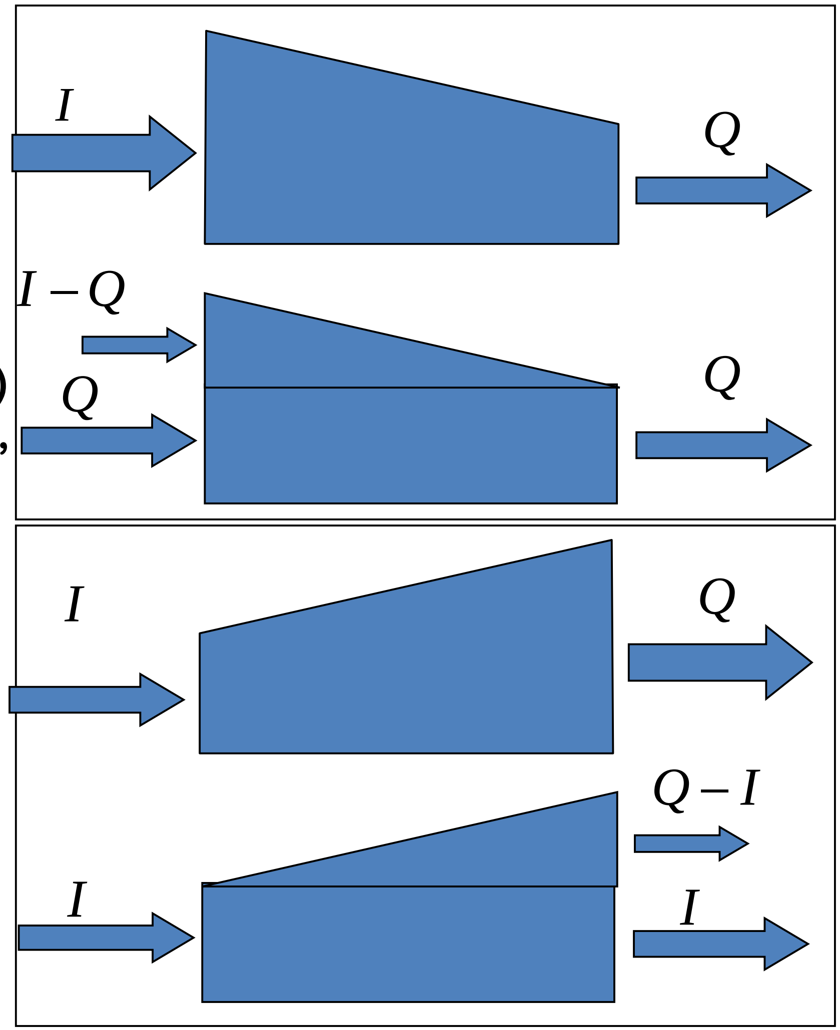
$$S_{\text{Prism}} = KQ$$

$$S_{\text{Wedge}} = KX(I-Q)$$

K = travel time of peak through the reach
X = weight on inflow versus outflow (0 ≤ X ≤ 0.5)
X = 0 ➔ Reservoir, storage depends on outflow, no wedge
X = 0.0 - 0.3 ➔ Natural stream

$$S = KQ + KX(I-Q)$$

$$\boxed{S = K[XI + (1-X)Q]}$$



Muskingum Method for Hydrologic River flood Routing

- Based $\frac{I_1 + I_2}{2} - \frac{(O_1 + O_2}{2} = \frac{S_2 - S_1}{\Delta t}$ and the

storage function $S = xKI^{m/d} + (1 - x)KO^{m/d}$

- Let m/d=1 and k=b/a, then we have

$$S = K[xI + (1 - x)O]$$

- Substituting into the routing equation gives

$$\frac{I_1 + I_2}{2} - \frac{O_1 + O_2}{2} = \frac{K[xI_1 + (1 - x)O_1] - K[xI_2 + (1 - x)O_2]}{\Delta t}$$

Muskingum Method

- Solving $$\frac{I_1 + I_2}{2} - \frac{O_1 + O_2}{2} = \frac{K[xI_1 + (1-x)O_1] - K[xI_2 + (1-x)O_2]}{\Delta t}$$

and rearranging yields

$$\mathrm{O_2 = C_o\ I_2 + C_1\ I_1 + C_2\ O_1}$$

Where $\mathrm{C_o = \dfrac{-KX + 0.5\Delta t}{K - KX + 0.5\Delta t}}$; $\mathrm{C_1 = \dfrac{KX + 0.5\Delta t}{K - KX + 0.5\Delta t}}$ and $\mathrm{C_2 = 1 - C_o - C_1}$

Note: K and Δt must have the same unit and initial estimate of O_1 must be specified as well as K and Δt .

Muskingum Routing Equation

$$O_2 = C_0 I_2 + C_1 I_1 + C_2 O_1$$

$$O_{t+1} = C_0 I_{t+1} + C_1 I_t + C_2 O_t$$

where C's are functions of x, K, Δt and sum to 1.0

Muskingum Equations

where

$$C_0 = (-Kx + 0.5\Delta t) / D$$

$$C_1 = (Kx + 0.5\Delta t) / D$$

$$C_2 = (K - Kx - 0.5\Delta t) / D$$

$$D = (K - Kx + 0.5\Delta t)$$

Repeat for O_3, O_4, O_5 and so on.

Estimating K

- K is estimated to be the travel time through the reach.
- This may pose somewhat of a difficulty, as the travel time will obviously change with flow.
- The question may arise as to whether the travel time should be estimated using the average flow, the peak flow, or some other flow.
- The travel time may be estimated using the kinematic travel time or a travel time based on Manning's equation.

Estimating X

- The value of X must be between 0.0 and 0.5.
- The parameter X may be thought of as a weighting coefficient for inflow and outflow.
- As inflow becomes less important, the value of X decreases.
- The lower limit of X is 0.0 and this would be indicative of a situation where inflow, I, has little or no effect on the storage.
- A reservoir is an example of this situation and it should be noted that attenuation would be the dominant process compared to translation.
- Values of X = 0.2 to 0.3 are the most common for natural streams; however, values of 0.4 to 0.5 may be calibrated for streams with little or no flood plains or storage effects.
- A value of X = 0.5 would represent equal weighting between inflow and outflow and would produce translation with little or no attenuation.

Estimating Muskingum Parameters, K and x

Graphical Method:

- Referring to the Muskingum Model, find X such that the plot of $XI_t + (1-X)O_t$ (m^3/s) vs S_t ($m^3/s.h$) behaves almost nearly as a single value curve. Assume value of x lies between 0 and 0.3.
- The corresponding slope is K.

Muskingum Routing Procedure

- Given (knowns): O_1; I_1, I_2, …; Δt; K; X
- Find (unknowns): O_2, O_3, O_4, …

- Procedure:

(a) Calculate C_o, C_1, and C_2

(b) Apply $O_{t+1} = C_o\, I_{t+1} + C_1\, I_t + C_2\, O_t$ starting from t=1, 2, … recursively.

Muskingum Notes :

- The method assumes a single stage-discharge relationship.
- In other words, for any given discharge, Q, there can be only one stage height.
- This assumption may not be entirely valid for certain flow situations.
- For instance, the friction slope on the rising side of a hydrograph for a given flow, Q, may be quite different than for the recession side of the hydrograph for the same given flow, Q.
- This causes an effect known as hysteresis, which can introduce errors into the storage assumptions of this method.

Muskingum Example Problem

• A portion of the inflow hydrograph to a reach of channel is given below. If the travel time is K=1 unit and the weighting factor is X=0.30, then find the outflow from the reach for the period shown below:

Time	Inflow	C_0I_2	C_1I_1	C_2O_1	Outflow
0	3				3
1	5				
2	10				
3	8				
4	6				
5	5				

Muskingum Example Problem

- The first step is to determine the coefficients in this problem.
- The calculations for each of the coefficients is given below:

$$C_0 = -\frac{Kx - 0.5\Delta t}{K - Kx + 0.5\Delta t}$$

C_0= - ((1*0.30) - (0.5*1)) / ((1-(1*0.30) + (0.5*1)) = 0.167

$$C_1 = \frac{Kx + 0.5\Delta t}{K - Kx + 0.5\Delta t}$$

C_1= ((1*0.30) + (0.5*1)) / ((1-(1*0.30) + (0.5*1)) = 0.667

Muskingum Example Problem

$$C_2 = \frac{K - Kx - 0.5\Delta t}{K - Kx + 0.5\Delta t}$$

C_2= (1- (1*0.30) - (0.5*1)) / ((1-(1*0.30) + (0.5*1)) = 0.167

Therefore the coefficients in this problem are:

- C_0 = 0.167
- C_1 = 0.667
- C_2 = 0.167

Muskingum Example Problem

- The three columns now can be calculated.
- $C_0I_2 = 0.167 * 5 = 0.835$
- $C_1I_1 = 0.667 * 3 = 2.00$
- $C_2O_1 = 0.167 * 3 = 0.501$

Time	Inflow	C_0I_2	C_1I_1	C_2O_1	Outflow
0	3	0.835	2.00	0.501	3
1	5				
2	10				
3	8				
4	6				
5	5				

Muskingum Example Problem

- Next the three columns are added to determine the outflow at time equal 1 hour.

- 0.835 + 2.00 + 0.501 = 3.34

Time	Inflow	C_0I_2	C_1I_1	C_2O_1	Outflow
0	3	0.835	2.00	0.501	3
1	5				3.34
2	10				
3	8				
4	6				
5	5				

Muskingum Example Problem

- This can be repeated until the table is complete and the outflow at each time step is known.

Time	Inflow	C_0I_2	C_1I_1	C_2O_1	Outflow
0	3	0.835	2.00	0.501	3
1	5	1.67	3.34	0.557	3.34
2	10	1.34	6.67	0.93	5.57
3	8	1.00	5.34	1.49	8.94
4	6	0.835	4.00	1.31	7.83
5	5		3.34	1.03	6.14

End of lecture 6